Binomial coefficients

1 Introduction – Counting

In these notes, we will discuss many properties of binomial coefficients and go over some topics where binomail coefficients naturally arise. We start by giving a definition.

Definition 1.1 (Binomial coefficient, combinatorial definition)**.** For non-negative integers $n \geq k \geq 0$ define a binomial coefficient $\binom{n}{k}$ (we read this as "n choose k") as the number of ways to pick k elements out of set of n elements (which is equal to 1 if k is zero, as we can only choose the empty set). If we do not have $n \geq k$ we set $\binom{n}{k} = 0$.

We can right away prove some basic properties about binomial coefficients.

Proposition 1.1 (Number of subsets of $\{1, \dots n\}$)**.** *The total number of subsets of set* $\{1, \dots, n\}$*, including an empy set is*

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof. By the definition of binomial coefficient, we have that the number of ways to pick a set of k elements out of $\{1, 2, \dots, n\}$ is $\binom{n}{k}$. Therefore, the number of ways to pick a subset of any size between $k = 0$ and $k = n$ is the sum $\sum_{k=0}^{n} \binom{n}{k}$. To see that the number of ways to pick a subset of any size is also 2^n, we notice that to pick a subset of $\{1, \dots, n\}$ we just need to decide if each of the elements between 1 and n belongs to the set we are picking or not. This means that for each of $1, 2, \dots, n$ we have two options, "in the set" or "not in the set", which gives $2 \times 2 \times 2 \dots \times 2 = 2^n$ subsets. □

Proposition 1.2. *We have the following equalities with binomial coefficients:*

1. *For* $n \geq k \geq 0$ *we have* $\binom{n}{k} = \binom{n}{n-k}$.
2. *For all* $n \geq k > 0$ *we have* $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
3. *For* $n \geq k \geq r \geq 0$ *we have* $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$.
4. *For* $a + b \geq r \geq 0, a, b \geq 0$ *we have* $\binom{a}{r}\binom{b}{0} + \binom{a}{r-1}\binom{b}{1} + \ldots + \binom{a}{0}\binom{b}{r} = \binom{a+b}{r}$.

Proof. 1. To pick k elements out of n one can first select $n - k$ elements and then pick the k non selected elements. The number of ways to select $n - k$ elements is $\binom{n}{n-k}$, giving the desired equality.

2. We pick k elements out of the set $\{1, \dots n\}$ by first deciding if n is in the set and then if it is, we pick $k-1$ elements from $\{1, \dots n-1\}$ and if it is not we pick k elements from $\{1, \dots n\}$. This means that the number of ways to pick k elements out of n is just the sum of $\binom{n-1}{k-1}$ (for the case when n is picked) and $\binom{n-1}{k}$ (for the case when n is not picked), and the result follows.

3. Let's count the number of ways to pick sets $X \subset Y \subset \{1, \dots n\}$ where $|X| = r$, $|Y| = k$. We can either first pick elements in Y in $\binom{n}{k}$ ways, and for each choice of Y we can pick elements in X in $\binom{k}{r}$ ways, which gives the left-hand side of the expression. Alternatively, we can pick X first in $\binom{n}{r}$ ways and then, for each choice of X, pick elements for $Y \setminus X$ in $\binom{n-r}{k-r}$ ways, which gives the right-hand side. This means that both the left-hand side and the right-hand side are counting the number of such choices of set pair (X, Y), and so the expressions need to be equal.

4. Suppose you want to make a fruit salad which has r fruits and you can choose from a bananas and b oranges. By definition, we can choose r fruits from the total of $a+b$ fruits in $\binom{a+b}{r}$ ways. On the other hand, we can first choose a number of bananas, s, and then take s out of a bananas and $r-s$ out of b oranges, which can be done in $\binom{a}{s}\binom{b}{r-s}$ ways and so summing over the choice of s gives the wanted identity (cases when $b < s$ or $a < s$ are still valid and the binomial coefficients are just 0 by definition).

□

Using ii) from Proposition 1.2 we see that if one makes a triangle of numbers where the left side and the right side have only 1s and all other values are recursively obtained by adding two values above them, we get that all numbers in the triangle are binomial coefficients.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

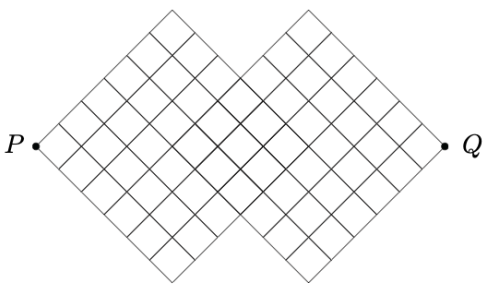
We notice that while the previous definition already allowed us to get many identities involving binomial coefficients, we still do not have any simple way of computing them. In how many ways can we pick a set of size $k > 0$ from a set of size $n > k$? Let's just consider a simple case where $n = 10$ (assume that the set we are picking from is $\{1, \dots, 10\}$) and $k = 2$ first. So, we need to pick $a \leq 10$, which can be done in 10 ways, and then we need

to pick $b \leq 10$ such that $b \neq a$, which can be done in 9 ways. This means that there are $10 \cdot 9$ ways to pick an ordered pair of distinct elements a and b, but the set $\{a, b\} = \{b, a\}$, so we are counting each set exactly twice, and the total number of sets is $\frac{10 \cdot 9}{2}$. Now for general n and k we have $n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$ ways to pick a list $a_1, \ldots, a_k$ of k distinct integers $\leq n$. How many times is each set counted by this product? Any permutation (reordering) of $a_1, \ldots, a_k$ gives exactly the same set, and the number of ways to reorder $a_1, \ldots, a_k$ is $k \cdot (k-1) \cdot \ldots \cdot 1$, as we have k options for which of the $a_1 \ldots, a_k$ will be in the first place, then $k-1$ for the second place etc. until we are left with one option for the final place. This means that the total number of ways to pick k elements out of set of size n is $\frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1} = \frac{n!}{(n-k)!k!}$ where $n! = n \cdot (n-1) \cdot \ldots \cdot 1$. This gives us the following algebraic definition of the binomial coefficient, which we have just proved is equivalent to Definition 1.1.

Definition 1.2 (Binomial coefficient, algebraic definition)**.** For non-negative integers $n \geq k$ define a binomial coefficient $\binom{n}{k}$ (we read this as "n choose k") by $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ (where we use the convention that $0! = 1$). If we do not have $n \geq k$ we set $\binom{n}{k} = 0$.

1.1 Questions

1. How many subsets of $\{1, 2, \ldots n\}$ have even size?
2. By interpreting both sides suitably, show the following identities:
   (a) $\binom{n}{k} + \binom{n-1}{k} + \ldots + \binom{k+1}{k} + \binom{k}{k} = \binom{n+1}{k+1}$
   (b) $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$.
3. A shop has n types of fruits, and they have more than r pieces of each of these n types. You want to make a salad which contains a total of r pieces of fruit (you are allowed to pick any number of fruits of each type, as long as the total number of pieces is r). Show that there is a total of $\binom{n+r-1}{r}$ different salads you could make.
4. In how many ways can 11 distinct birds be placed into 3 identical cages, such that each cage contains at least 3 birds?
5. How many shortest possible paths are there between points P and Q, where movement is allowed only along the edges of the square grid shown in the diagram?



6. For $0 \leq k \leq n$ prove that the following identity holds:

$$\sum_{i=0}^{k} \binom{n+1}{i} + \sum_{i=0}^{n-k} 2^i \binom{n-i}{k} = 2^{n+1}$$

7. Let A and B be non-empty subsets of the set $\{1, 2, \ldots, n\}$. We define $A < B$ if every element of A is less than every element of B. Prove that the number of pairs of non-empty subsets (A, B) such that $A < B$ is equal to: $(n-2) \cdot 2^{n-1} + 1$

2 Divisibility

We will now explore some divisibility properties of binomial coefficients. First, recall that an integer greater than 1 is a prime if it is only divisible by itself and 1. Recall also that a natural number has a unique prime factorisation, which means that it can be expressed in a unique way as a product of primes (unique up to reordering of these primes). So we can write any natural number n as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

where an integer $k \geq 0$, p_i are distinct prime numbers and α_i are natural numbers.

We say that $a \equiv b \pmod{k}$ if $k \mid a - b$ (we read this as "a is congruent to b mod k"), which means k divides $a - b$, i.e. that $\frac{a-b}{k}$ is an integer (either positive or negative). This is a standard notation in number theory which helps shorten many expressions related to divisibility.

Here are two simple examples where we can use this notation:

Let $n = a_d \ldots a_1 a_0$ be a number written in base 10. We want to show that the remainder of dividing n by 9 is the same as the remainder of dividing $a_0 + a_1 + a_2 + \ldots + a_d$ by 9, which would also imply that a number is divisible by 9 if and only if the sum of its digits is. In other words, we want to show that $n = \sum_{i=0}^{d} a_i \pmod{9}$. We have

$$a_d \ldots a_1 a_0 = \sum_{i=0}^{d} a_i \cdot 10^i \equiv \sum_{i=0}^{d} a_i \cdot 1^i \equiv \sum_{i=0}^{d} a_i \pmod{9}$$

where we have used that $10 \equiv 1 \pmod{9}$ and this proves the desired statement. In this example, we have used that if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ then $ac \equiv bd \pmod{k}$ and $a + b \equiv c + d \pmod{k}$, which are properties one could easily verify (and you are encouraged to do this!).

Let $n = 2k + 1$ be an odd number. We want to show that the residue of dividing n^2 by 8 is 1. Indeed, we have

$$n^2 \equiv (2k+1)^2 \equiv 4k^2 + 2 \cdot 2k + 1 \equiv 4k(k+1) + 1 \equiv 1 \pmod{8}$$

where we have used that $8 \mid 4k(k+1)$ as, regardless of the parity of k, one of k and $k+1$ is even.

Another useful property of modular arithmetic is that for any integers a, b, c, d and any natural number k which is coprime to each one of a, b, c, d[1] we have that if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ and if $\frac{a}{c}$ and $\frac{b}{d}$ are integers then $\frac{a}{c} \equiv \frac{b}{d} \pmod{k}$. (Check this as well!)

The following are a few questions which try to investigate some divisibility properties of binomial coefficients.

2.1 Questions

1. Show that for any prime number p and $k < p$ we have $\binom{p-1}{k} = (-1)^k \pmod{p}$.
2. Let $0 < k < n+1$ and let p be a prime number. Prove that if $p \nmid \binom{n}{k}$, then $p \nmid \binom{n+1}{k}$, except in the case when $p \mid (n+1)$.
3. Show that for $k \leq \frac{p-1}{2}$, where p is a prime, we have $\binom{2k}{k} \equiv (-4)^k \binom{\frac{p-1}{2}}{k} \pmod{p}$.
4. Let
$$A = (2011+i)^{2010} + (2011-i)^{2010}$$
   (a) Prove that A is an integer.
   (b) Determine the remainder when A is divided by 101.
5. Let p be a prime number. Prove that the exponent of p in the prime factorisation of the binomial coefficient $\binom{n}{m}$ where $0 \leq m \leq n$, is equal to the number of carries that occur when adding m and $n-m$ in base p.
6. Let p be a prime number. Let n and m be non-negative integers. Write their base-p expansions as: $n = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k$, and $m = m_0 + m_1 p + m_2 p^2 + \cdots + m_k p^k$, where $0 \leq n_i, m_i < p$. Show that:
$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}.$$
7. Prove that the number of odd binomial coefficients in the n-th row of Pascal's triangle is equal to 2^r, where r is the number of 1's in the binary expansion of n.

[1] This means that k and a are coprime, k and b are coprime, k and c are coprime and k and d are coprime. Integers m and n are coprime if there is no prime p which divides both of them.

3 Probability

For a random variable X taking values in non-negative integers, we first define the mean as

$$\mu_X = \mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k)$$

and the variance as

$$\sigma_X^2 = \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

One connection between binomial coefficients and probability is, of course, the binomial distribution, and we will not calculate its mean and variance.

Suppose we toss a coin n times, each with probability p of getting heads. Then the probability of a specific sequence like $\mathrm{HHTT}\cdots\mathrm{T}$ is:

$$\mathbb{P}(\mathrm{HHTT}\cdots\mathrm{T}) = p \cdot p \cdot (1-p)\cdots(1-p) = p^2(1-p)^{n-2}.$$

Since there are $\binom{n}{2}$ ways to choose 2 positions out of n to be heads, the total probability of getting exactly 2 heads is:

$$\mathbb{P}(\text{two heads}) = \binom{n}{2} p^2(1-p)^{n-2}.$$

In general, the probability of getting exactly k heads is:

$$\mathbb{P}(k \text{ heads}) = \binom{n}{k} p^k(1-p)^{n-k}.$$

This is known as the binomial distribution, denoted often by $B(n,p)$.

Let $X \sim B(n,p)$ be a binomial random variable, i.e. the number of heads in the above example.

Let $X = X_1 + X_2 + \cdots + X_n$, where each $X_i \sim B(1,p)$ is an random variable which is 1 if ith toss is heads and 0 otherwise. Then, as expectation is linear (check this if you have not seen it!),

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \cdots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} p = np.$$

By definition, the expectation of X is

$$\mathbb{E}[X] = \sum_{k=0}^{n} k\mathbb{P}(X = k) = \sum_{k=0}^{n} k\binom{n}{k} p^k(1-p)^{n-k}$$

which means that we have proved that

$$\sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} = np$$

has to hold. (You are encouraged to find an algebraic proof of this!)

To find the variance, consider

$$\begin{aligned}\mathbb{E}[X(X-1)] &= \sum_{r=0}^{n} r(r-1)\binom{n}{r}p^r(1-p)^{n-r} \\ &= n(n-1)p^2\sum_{r=2}^{n}\binom{n-2}{r-2}p^{r-2}(1-p)^{(n-2)-(r-2)} = n(n-1)p^2\end{aligned}$$

where the sum became 1 since it is the sum of all probabilities of a binomial $B(n-2,p)$. So, $\mathbb{E}[X^2] = n(n-1)p^2 + \mathbb{E}[X] = n(n-1)p^2 + np$. So,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = np(1-p).$$

The following are a few probability questions in which binomial coefficients might appear as part of the solution.

3.1 Questions

1. A full deck of 52 cards is divided in half at random. Find an expression for the probability that each half contains the same number of red and black cards.

2. Let $X \sim B(n,p)$. Show that $\mathbb{P}(X \text{ is even}) = \frac{1}{2}\left(1 + (1-2p)^n\right)$.

3. Mary tosses two coins and John tosses one coin. What is the probability that Mary gets more heads than John? Answer the same question if Mary tosses three coins and John tosses two. What is the probability when Mary tosses $n+1$ coins, and John tosses n?

4. Suppose that n balls are tossed independently and at random into n boxes. What is the probability that exactly one box is empty?

5. A rabbit is hopping on the integer square grid, starting from $(0,0)$, by making one step to the left, right, up or down every minute (i.e. if the current coordinates are (a,b), for $a,b \in \mathbb{Z}$ in the next minute rabbit will be on one of the coordinates $(a-1,b),(a+1,b),(a,b+1)$ or $(a,b-1)$).

   (a) What is the number of possible locations of the rabbit after 10 minutes?

   (b) If at each step the rabbit makes a jump by choosing each direction equally likely, i.e. with probability $\frac{1}{4}$, what is the probability that after 10 minutes the rabbit is at location $(5,5)$? Let $p_{(a,b)}$ be the probability that the rabbit is at location (a,b) after 10 minutes. Find the values of $p_{(4,6)}$, $p_{(3,6)}$ and $p_{(3,3)}$.

6. What is the probability that a random non-decreasing function $\{1, ..., k\} \to \{1, ..., n\}$ is increasing?

7. A simple random walk on $\mathbb{Z}$ is a process where a particle moves on integers by $+1$ or -1 with equal probability, starting from 0. In other words, the location of the particle at time n is $S_n = \sum_{i=1}^{n} X_i$ where X_i are independent and take values $+1$ or -1 with probability $\frac{1}{2}$ each. Find $\mathbb{P}(S_n = k)$ for all integers $n \geq k \geq -n$.

4 Polynomials and generating functions

By a real polynomial $P(x)$ we just mean the expressions of the form $a_n \cdot x^n + \ldots a_1 \cdot x + a_0$ where $n \geq 0$ is an integer and where $a_0, \ldots, a_n$ are real numbers. The polynomials could also be complex, integer, or rational polynomials and then $a_0, \ldots, a_n$ would be complex, integer or rational numbers, respectively.

Binomial coefficients appear naturally in many polynomial expressions, and we start with the most famous one, known as the binomial theorem.

Theorem 4.1 (Binomial theorem)**.** *For integer $n \geq 0$ we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. We are multiplying n brackets, each of them is $(x + y)$. To get the terms of the form $x^k y^{n-k}$, we need to pick x from k and y from the remaining $n - k$ brackets. By the definition of the binomial coefficient, we have that $\binom{n}{k}$ is the number of ways to pick k brackets from which we take x, so the wanted expression follows. □

Generating functions are expressions similar to polynomials, except that we allow the sum to be infinite. We can define them simply as follows.

Definition 4.1 (Generating function)**.** For a sequence $(a_n)_{n\geq 0}$ of complex numbers, we define a generating function corresponding to this sequence (or generating function of this sequence) to be a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

From the above definition, we see that Polynomials are generating functions themselves, just by setting $a_m = 0$ for all $m > n$ and some integer n. As generating functions are infinite sums, they might not converge for all values of x, and in fact, it could happen that they only converge for $x = 0$. However, we will not bother ourselves much with the question of convergence; we will simply think of a generating function as an algebraic expression which we do not necessarily want to evaluate at any particular number x.

If $A(x), B(x)$ and $C(x)$ are a generating function of seqences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ then we have:

1) $C(x) = \lambda A(x) + \mu B(x)$ is a generating function of sequence $c_n = \lambda a_n + \mu b_n$

2) $C(x) = A(x)B(x) = B(x)A(x)$ is a generating function of sequence $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

3) $A(x)(B(x)C(x)) = (A(x)B(x))C(x)$ i.e. we have associativity

We can notice that property two gives us that $(1 + x + x^2 + \ldots)(1 - x) = 1 + 0x + 0x^2 + \ldots = 1$. So it might be useful to think of $\sum_{n\geq 0} x^n$ as $\frac{1}{(1-x)}$. In general, we have the following definition.

Definition 4.2. A reciprocial of a generating function $A(x)$ is a generating function $B(x)$ such that $A(x)B(x) = 1$. We write $A(x) = \frac{1}{B(x)}$.

For a generating function $A(x)$ we say that a first derivative of $A(x)$ which we denote by $A'(x)$ is a generating function of sequence $(a_1, 2a_2, 3a_3 \ldots) = ((n+1)a_{n+1})_{n\geq 0}$, i.e.

$$A'(x) = \left(\sum_{n\geq 0} a_n x^n\right)' = \sum_{n\geq 1} n a_n x^{n-1}.$$

We define the kth derivative of $A(x)$, denoted by $A^{(k)}(x)$ recursively by

$$A^{(k)}(x) = \left(A^{(k-1)}(x)\right)'.$$

We now see an example where binomial coefficients appear in some simple manipulation with generating functions.

Proposition 4.1. *Let $A(x)$ and $B(x)$ be two generating functions. The nth derivative of their product satisfies*

$$(A(x) \cdot B(x))^{(n)} = \sum_{i=0}^{n} \binom{n}{i} A(x)^{(i)} B(x)^{(n-i)}.$$

Proof. To figure out if the two generating functions are the same, we want to work out if the coefficients on both sides next to the power x^k are the same for all values of k. On the left-hand side, using the definition of differentiation and property 2), we have next to x^k coefficient: $(k+1)\cdot(k+2)\ldots\cdot(k+n)\cdot\sum_{j=0}^{k+n} a_j b_{n+k-j}$. The expression on the right-hand side is somewhat more complicated, and we could try to directly check that it is the same. However, we could try first to think of a better way of doing this. The expression is easy to check if n=1, as $(A(x)B(x))' = A'(x)B(x) + A(x)B'(x)$. So maybe we can try to do induction? We know that the basis of induction holds, so what about the induction step? If $(A(x)\cdot B(x))^{(n)} = \sum_{i=0}^{n}\binom{n}{i}A(x)^{(i)}B(x)^{(n-i)}$ does the statement hold for n replaced by $n+1$? We can see that $\left(\sum_{i=0}^{n}\binom{n}{i}A(x)^{(i)}B(x)^{(n-i)}\right)' = \sum_{i=0}^{n+1}(\binom{n}{i}+\binom{n}{i-1})A(x)^{(i)}B(x)^{(n+1-i)}$ and from before we know that this is same as $\sum_{i=0}^{n+1}\binom{n+1}{i}A(x)^{(i)}B(x)^{(n+1-i)}$ and so the proof by induction follows. □

Consider any function F which is differentiable any number of times (i.e. infinitely differentiable) at 0. We can identify it with a generating function of $\left(\frac{F(x)^{(k)}|_{x=0}}{k!}\right)_{k\geq 0}$. This expansion of functions into power series is known as the Taylor expansion. As we think of generating functions as algebraic objects, i.e. we do not worry about convergence for any values of x, we can just think of Taylor expansion as a definition for the functions we are considering. For example, we have that

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \ldots.$$

This is how we can also make sense of the binomial theorem for any real power:

Theorem 4.2 (Binomial Theorem)**.** *For $n \in \mathbb{N}$ we have $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$. If $\alpha \in \mathbb{R}$ we have that $(1+x)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} x^i$, where we here define $\binom{\alpha}{i} = \frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!}$.*

We can notice that if $i > \alpha > 0$ and α is an integer, then the numerator in the above definition of $\binom{\alpha}{i}$ is 0 and therefore $\binom{\alpha}{i} = 0$. We can also notice that if $k > 0$ and $m > 0$ are integer, then

$$\binom{-k}{m} = \frac{-k(-k-1)\ldots(-k-m+1)}{m!} = (-1)^m \binom{k+m-1}{m}.$$

We now see an example of how one can use generating functions (or polynomials) to count things.

Question 4.1. In how many ways can we make a fruit salad of apples, bananas, oranges and pears consisting of n fruits, in which the number of apples is even, the number of bananas is a multiple of 5, there are at most 4 oranges, and there is at most one pear?

Let's make generating functions for apples, $A(x)$, bananas, $B(x)$, oranges, $O(x)$ and pears, $P(x)$. By this we mean the functions which would have coefficients with x^k either 0 or 1, where 0 will mean that having k objects of this type is not possible, while 1 would mean that it is possible. So we have:

$$\begin{aligned} A(x) &= 1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x^2} \\ B(x) &= 1 + x^5 + x^{10} + x^{15} + \ldots = \frac{1}{1-x^5} \\ O(x) &= 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x} \\ P(x) &= 1 + x \end{aligned}$$

We can compute

$$A(x)B(x)O(x)P(x) = \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = (1 + x + x^2 + \ldots)' = \sum_{n=0}^{\infty} (n+1)x^n$$

What do the coefficients of $A(x)B(x)O(x)P(x)$ represent? The coefficient of x^k was obtained by multiplying the coefficients a_i, b_j, o_ℓ and $p_{k-i-j-\ell}$ and summing this over all suitable i, j, ℓ. The coefficients a_i, b_j, o_ℓ and $p_{k-i-j-\ell}$ represent the number of ways to make a salad with i apples, j bananas, ℓ oranges and $k-i-j-\ell$ pears, and so their product is the number of ways to make a salad with k fruits in total, which has i apples, j bananas, ℓ oranges and when this is sumed over all i, j and ℓ this gives the number of ways to make a salad with k fruits. This means that the coefficients of $A(x)B(x)O(x)P(x)$ represent the number of ways to make a fruit salad with k fruits, and so the number of ways to make a salad with n fruits is $n+1$.

4.1 Questions

1. Factorise $x^n - y^n$ and $x^{2n+1} + y^{2n+1}$.

2. Evaluate the following sums:

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} \quad \text{and} \quad \sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k}.$$

   Deduce that:

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

   [Hint: You might want to integrate a certain polynomial.]

3. Show that:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r}^2 = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{n/2}\binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases}$$

4. For $n \geq m \geq 0$ define $a_m = \sum_{k=m}^{n} (-1)^k \binom{n}{k}\binom{k}{m}$. Show that $a_m = 0$ for $m < n$ and $a_n = (-1)^n$.

5. Let S be the set of sequences of length 2018 whose terms are in $\{1, 2, 3, 4, 5, 6, 10\}$ and sum up to 3860. Prove that the cardinality of S is at most $2^{3860}(\frac{2018}{2048})^{2018}$.

6. Find the sequence with the generating function equal to $\frac{1}{\sqrt{1-4x}}$ and deduce that for all integers n

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$$

7. For an integer n, the Catalan number C_n is defined to be the number of sequences of n open and n closed brackets, which satisfy that when the sequence is read from left to right, there are at least as many open brackets as closed ones at any point. For example, $C_3 = 5$, as valid configurations of brackets are ()()();(())();()(());((())); (()()), while ())(() would not be valid, as up to the third bracket there are more closed than open brackets. First show that Catalan numbers satisfy $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ and then show that $C_n = \frac{1}{n+1}\binom{2n}{n}$.